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Boundary bound states and boundary bootstrap in the sine–Gordon model with Dirichlet boundary conditions

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Abstract. We present a complete study of boundary bound states and related boundary S-matrices for the sine–Gordon model with Dirichlet boundary conditions. Our approach is based partly on the bootstrap procedure and partly on the explicit solution of the inhomogeneous XXZ model with boundary magnetic field and solution of the boundary Thirring model. We identify boundary bound states with new 'boundary strings' in the Bethe ansatz. The boundary energy is also computed.

1. Introduction

The sine–Gordon model with a boundary interaction preserving integrability (which we shall call the boundary sine–Gordon model) is of theoretical as well as practical interest. In particular, it exhibits relations with the theory of Jack symmetric functions [1] and has applications to dissipative quantum mechanics [2] and impurity problems in 1D strongly correlated electron gas [3].

In the seminal work [4] it appears clearly that this problem presents an extremely rich structure of boundary bound states, which has been partly explored in [5]. Our first purpose, here, is to study this structure further in the particular case of Dirichlet boundary conditions, which is the model

$$\mathcal{L}_{\mathrm{SG}} = \tfrac{1}{2}\int_0^\infty \left[(\partial_t\varphi)^2 - (\partial_x\varphi)^2 + \frac{m_0^2}{\beta^2}\cos\beta\varphi\right]\mathrm{d}x \tag{1.1}$$

with a fixed value of the field at the boundary, $\varphi(x=0,t)=\varphi_0$.

Also, the consideration of boundary problems poses interesting challenges from the point of view of lattice models, or in this case lattice regularizations of (1.1). In [6] and also in [7] it was shown, in particular, how to derive the S-matrices of [4] from the Bethe ansatz. Our second purpose is to complete these studies by investigating which new types of strings correspond to boundary bound states, and by also deriving the set of S-matrices necessary to close the bootstrap. Note that lattice regularizations are useful when defining what one means by creating a bound state at the boundary. Indeed, some bound states (showing up as the poles of S-matrices) have no straightforward interpretation, and although they are easy to study formally using the Yang Baxter equation and the bootstrap, their meaning in the field theory is unclear.

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In section 2, we consider the bootstrap problem directly in the continuum theory. We identify boundary bound states and we compute the related boundary S-matrices. In section 3, we write the Bethe ansatz equations for the inhomogeneous six-vertex model with boundary magnetic field, which is believed [6] to be a regularization of (1.1). We show that these equations are also the bare equations for the Thirring model with $U(1)$-preserving boundary interaction, which is the fermionized version of (1.1). In section 4, we discuss in detail new solutions ('boundary strings') to the Bethe ansatz equations made possible by the appearance of boundary terms. In section 5, we study the physical properties of the model, in particular the masses and S-matrices corresponding to these boundary strings, and we partially complete the identification with the bootstrap results of section 2. Several remarks, in particular formulae for the boundary energy of the boundary sine–Gordon model, are collected in the conclusion.

2. Boundary bootstrap results

2.1. Solving the boundary bootstrap equations

The S-matrices for the scattering of a soliton (P^+) and an anti-soliton (P^-) on the ground state $|0\rangle_B$ of the sine–Gordon model with Dirichlet boundary conditions (1.1) were obtained in [4]:

$$P^{\pm}(\theta) = \cos(\xi \pm \lambda u) R_0(u) R_1(u, \xi) \tag{2.1}$$

where $\theta = iu$ is the rapidity, $\xi = 4\pi\varphi_0/\beta$ and $\lambda = 8\pi/\beta^2 - 1$. The explicit form of R_0, R_1 is rather cumbersome and can be found in [4]. Since the theory is invariant under the simultaneous transformations $\xi \to -\xi$, and soliton→anti-soliton, hereafter we choose ξ to be a generic number in the interval $0 < \xi < 4\pi^2/\beta^2$ (see the discussion in [8] about the value of the upper bound).

The function R_0 contains poles in the physical strip $0 < \mathrm{Im}\,\theta < \pi/2$ located at $u = n\pi/2\lambda$, $n = 1, 2, \ldots < \lambda$. These poles come from the corresponding breather pole in the soliton–antisoliton bulk scattering, and should not be interpreted as boundary bound states [4].

When $\xi > \pi/2$, the function $P^+(\theta)$ has additional poles in the physical strip, located at $u = v_n$ with

$$0 < v_n = \frac{\xi}{\lambda} - \frac{2n+1}{2\lambda}\pi < \frac{\pi}{2} \tag{2.2}$$

($n = 0, 1, 2, \ldots$) corresponding to a first set of boundary bound states which we denote by $|\beta_n\rangle$, with masses

$$m_n = m \cos v_n = m \cos\left(\frac{\xi}{\lambda} - \frac{2n+1}{2\lambda}\pi\right) \tag{2.3}$$

where m is the soliton mass. These bound states are easy to interpret [4, 8]. For $0 < \varphi_0 < \pi/\beta$ the ground state of the theory is characterized by the asymptotic behaviour $\varphi \to 0$ as $x \to \infty$, but other states, whose energy differs from the ground state by a boundary term only, can be obtained with $\varphi \to$ {a multiple of $2\pi/\beta$} as $x \to \infty$. Since the β_n appear as bound states for soliton scattering, they all have the same topological charge as the soliton, which we take to be equal to unity by convention, so they are all associated with the same classical solution, a soliton sitting next to the boundary and performing a motion periodic in time ('breathing'), with $\varphi(x = 0) = \varphi_0$ and $\varphi \to 2\pi/\beta$ as $x \to \infty$ [8].

To deduce the scattering matrices on the boundary bound states we use the 'boundary bootstrap equations' as given in [4]. We assume that these S-matrices are diagonal, which is true if all the boundary bound states have different energies. In this case the bootstrap equations read:

$$R_{\beta}^{b}(\theta) = \sum_{c,d} R_{\alpha}^{d}(\theta) S_{cd}^{ab}(\theta + \mathrm{i}v_{\alpha a}^{\beta}) S_{ba}^{dc}(\theta - \mathrm{i}v_{\alpha a}^{\beta}). \tag{2.4}$$

These equations allow us to find the scattering matrix of any particle b on the boundary bound state β provided that the latter appears as a virtual state in the scattering of the particle a on the boundary state α. The masses of the corresponding boundary states are related through

$$m_{\beta} = m_{\alpha} + m_{a} \cos v_{\alpha a}^{\beta} \tag{2.5}$$

where $\mathrm{i}v_{\alpha a}^{\beta}$ denotes the position of the pole, corresponding to the bound state β.

Let β_n stand for the nth boundary bound state corresponding to the pole v_n in P^{+} (2.2). Then (2.4) gives

$$P_{\beta_n}^{+}(\theta) = P^{+}(\theta) a(\theta - \mathrm{i}v_n) a(\theta + \mathrm{i}v_n) \tag{2.6}$$

$$P_{\beta_n}^{-}(\theta) = b(\theta - \mathrm{i}v_n) b(\theta + \mathrm{i}v_n) P^{-}(\theta) + c(\theta - \mathrm{i}v_n) c(\theta + \mathrm{i}v_n) P^{+}(\theta) \tag{2.7}$$

where the well known bulk S-matrix elements $a(\theta) = S_{++}^{++} = S_{--}^{--}$ (kink–kink scattering), $b(\theta) = S_{+-}^{+-} = S_{-+}^{-+}$ (kink–anti-kink transmission), and $c(\theta) = S_{+-}^{-+} = S_{-+}^{+-}$ (kink–anti-kink reflection) can be found in [9].

It is easy to check that the matrix elements (2.6)–(2.7) satisfy general requirements for the boundary S-matrices, such as boundary unitarity and boundary crossing-symmetry conditions [4], e.g.

$$P_{\beta_n}^{-}\left(\frac{\mathrm{i}\pi}{2} - \theta\right) = b(2\theta) P_{\beta_n}^{+}\left(\frac{\mathrm{i}\pi}{2} + \theta\right) + c(2\theta) P_{\beta_n}^{-}\left(\frac{\mathrm{i}\pi}{2} + \theta\right)$$

$$P_{\beta_n}^{\pm}(\theta) P_{\beta_n}^{\pm}(-\theta) = 1.$$

Finally, we obtain from (2.6)–(2.7) by direct calculation

$$P_{\beta_n}^{+}(\theta) = \frac{\cos(\xi - \lambda\pi - \mathrm{i}\lambda\theta)}{\cos(\xi - \lambda\pi + \mathrm{i}\lambda\theta)} P_{\beta_n}^{-}(\theta). \tag{2.8}$$

Hence the boundary Yang Baxter equation is satisfied, since the ratio of the above two amplitudes has a form similar to (2.1) with $\xi \to \xi - \lambda\pi$, ξ being a free parameter.

The analytic structure of $P_{\beta_n}^{\pm}(\theta)$ is as follows. The function $P_{\beta_n}^{+}(\theta)$ has simple poles in the physical strip located at $u = (\xi/\lambda) + ((2N+1)/2\lambda)\pi$, $N = 0, 1, 2, \ldots$, and at $u = v_n$. It has double poles at $u = \mathrm{i}v_n + \mathrm{i}k\pi/\lambda$, $k = 1, 2, \ldots, n$. The function $P_{\beta_n}^{-}(\theta)$ possesses in the physical strip the same singularities as $P_{\beta_n}^{+}(\theta)$ plus the set of simple poles at $u = w_N$ with

$$w_N = \pi - \frac{\xi}{\lambda} - \frac{2N-1}{2\lambda}\pi \qquad \lambda + \frac{1}{2} - \frac{\xi}{\pi} > N > \frac{\lambda+1}{2} - \frac{\xi}{\pi}. \tag{2.9}$$

Interpreting these poles in terms of boundary bound states requires some care. First, due to the relation (2.4), one sees that if β appears as a boundary bound state for scattering of a on α, then the poles of the amplitude for scattering of b on α are also in general poles of the amplitude for scattering of b on β. It seems unlikely that these poles correspond to new bound states, although in our case they would have a natural physical meaning, for example one could try to associate them with classical solutions where $\varphi \to 4\pi/\beta$ as $x \to \infty$. Indeed there are strong constraints coming from statistics that we should not forget. For instance, at the free fermion point $\beta^2 = 4\pi$ there is a bound state β_1, but although $P^+_{\beta_1}$ has again a pole at β_1, the state of mass $2m_{\beta_1}$ is not allowed from the Pauli exclusion principle, as can easily be checked on the direct solution of the model (see section 3.3). Therefore, we take the point of view that the poles already present in the scattering on an 'empty boundary' are 'redundant'. The only poles we interpret as new boundary bound states are (2.9) (the additional poles in $P^+_{\beta_n}$ are related to them be crossing). We denote these boundary bound states $|\delta_{n,N}\rangle$, and their masses, according to (2.5) and (2.3), are given by

$$m_{n,N} = m(\cos v_n + \cos w_N) = m\cos\left(\frac{\xi}{\lambda} - \frac{2n+1}{2\lambda}\pi\right) - m\cos\left(\frac{\xi}{\lambda} + \frac{2N-1}{2\lambda}\pi\right)$$
$$= m^b_{N+n}\sin\left(\frac{\xi}{\lambda} + \frac{N-n-1}{2\lambda}\pi\right) \tag{2.10}$$

where $m^b_p = 2m\sin(p\pi/2\lambda)$ is the mass of the pth breather, $p = 1, 2, \ldots < \lambda$.

To understand the physical meaning of these new boundary bound states it is helpful to consider the semi-classical limit $\lambda \to \infty$ of the sine–Gordon model. As discussed above, the boundary bound states β_n, corresponding to (2.2), are associated with solutions where a soliton is sitting next to the boundary and 'breathing'. An incoming anti-soliton can couple to this soliton, and together they form a breather sitting next to the boundary and performing again some (rather complicated) motion periodic in time†. The WKB quantization of this solution [10] should lead to $|\delta_{n,N}\rangle$. The topological charge of the states $|\delta_{n,N}\rangle$ is equal to zero in our units, or, equivalently, to the charge of a free breather in the theory (1.1).

One can, in principle, continue to solve the bootstrap equations (2.4) recursively. For example, for the scattering of solitons or anti-solitons on the boundary bound states $|\delta_{n,N}\rangle$ (2.9) one obtains the following S-matrices:

$$P^-_{\delta_{n,N}}(\theta) = P^-_{\beta_n}(\theta)a(\theta - \mathrm{i}w_N)a(\theta + \mathrm{i}w_N) \tag{2.11}$$

$$P^+_{\delta_{n,N}} = \frac{\cos(\xi - \mathrm{i}\lambda\theta)}{\cos(\xi + \mathrm{i}\lambda\theta)}P^-_{\delta_{n,N}}. \tag{2.12}$$

$P^-_{\delta_{n,N}}$ has only one simple pole in the physical strip at $u = w_N$, while $P^+_{\delta_{n,N}}$ also has simple poles at $u = v_k$, $k = n+1, n+2, \ldots, [\xi/\lambda - \frac{1}{2}]$. According to the discussion below (2.9), we do not consider these poles as associated with new boundary bound states. Therefore, the boundary bootstrap is closed for solitons and anti-solitons in the sense that further recursion will not generate new boundary bound states.

So far we have obtained two sets of boundary bound states (2.3) and (2.10) by considering all the poles in the physical strip of amplitudes for scattering a soliton and anti-soliton on a boundary with or without a boundary bound state. Of course we should also consider the scattering of breathers off the boundary. The scattering of breathers on the 'empty' boundary was studied in [5], and we refer the reader to this work for the explicit boundary S-matrices. By interpreting the poles of the amplitudes in [5] as boundary bound states, we find a spectrum of masses that look like (2.10) but with a slightly different range of parameters. Therefore, considering scattering of breathers off a boundary with a bound state does not give rise to any new poles besides (2.2) and (2.9), with in the latter case an extended range of values of N (for simplicity we do not give the relevant boundary S-matrices here). Therefore the complete boundary bootstrap is closed in principle.

† To compute this solution explicitly requires the use of a bulk five-soliton configuration [8], an expression which is very cumbersome.

2.2. Integral representations of various S-matrices

For comparison with results obtained from regularizations of the sine–Gordon model it is useful to write integral representations of the boundary S-matrices (2.1), (2.6), and (2.7) using the well known formula

$$\log\Gamma(z) = \int_0^{\infty}\frac{\mathrm{d}x}{x}\mathrm{e}^{-x}\left[z-1+\frac{\mathrm{e}^{-(z-1)x}-1}{1-\mathrm{e}^{-x}}\right] \qquad \mathrm{Re}\, z>0. \tag{2.13}$$

Suppose first that $1<2\xi/\pi<\lambda+1$ *and denote*

$$n_* = \left[\frac{\xi}{\pi}-\frac{1}{2}\right] \tag{2.14}$$

where the square brackets mean the integer part of the number. For such values of ξ there are n_*+1 poles (2.2) in the physical strip, i.e. the spectrum of excitations contains boundary bound states. Correspondingly, there is a finite number of Γ-functions in (2.1), (2.6), and (2.7) whose arguments have negative real part so that formula (2.13) is not applicable. Treating such Γ-functions separately, we obtain the following results:

$$\begin{aligned}-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\theta}\log\left[\frac{P^+(\theta)}{R_0(\theta)}\right] &= \frac{2\lambda}{\pi}\int_{-\infty}^{+\infty}\mathrm{d}x\,\cos\left(\frac{2\lambda\theta x}{\pi}\right)\\ &\quad\times\left[\frac{\sinh(2\xi/\pi-2n_*-2)x}{\sinh x}+\frac{\sinh(\lambda-2\xi/\pi)x}{2\sinh x\cosh\lambda x}\right]\end{aligned} \tag{2.15}$$

$$\begin{aligned}-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\theta}\log\left[\frac{P^+_{\beta_n}(\theta)}{R_0(\theta)}\right] &= \frac{2\lambda}{\pi}\int_{-\infty}^{+\infty}\mathrm{d}x\,\cos\left(\frac{2\lambda\theta x}{\pi}\right)\\ &\quad\times\frac{\sinh(\lambda-2\xi/\pi)x-2\cosh x\sinh(\lambda+1+2n-\xi/\pi)x}{2\sinh x\cosh\lambda x}\end{aligned} \tag{2.16}$$

$$\begin{aligned}-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\theta}\log\left[\frac{P^-_{\beta_n}(\theta)}{R_0(\theta)}\right] &= \frac{2\lambda}{\pi}\int_{-\infty}^{+\infty}\mathrm{d}x\,\cos\left(\frac{2\lambda\theta x}{\pi}\right)\left[\frac{\sinh(2n_*+2-2\xi/\pi)x}{\sinh x}\right.\\ &\quad\left.+\frac{\sinh(\lambda-2\xi/\pi)x-2\cosh x\sinh(\lambda+1+2n-2\xi/\pi)x}{2\sinh x\cosh\lambda x}\right].\end{aligned} \tag{2.17}$$

In the derivation of the analogous representation for P^- there are no subtleties because the 'dangerous' Γ-functions cancel. We get

$$-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\theta}\log\left[\frac{P^-(\theta)}{R_0(\theta)}\right] = \frac{2\lambda}{\pi}\int_{-\infty}^{+\infty}\mathrm{d}x\,\cos\left(\frac{2\lambda\theta x}{\pi}\right)\frac{\sinh(\lambda-2\xi/\pi)x}{2\sinh x\cosh\lambda x}. \tag{2.18}$$

In the region $0<2\xi/\pi<1$, where there are no poles and no boundary bound states in the spectrum, formula (2.18) is valid too. The expression for P^+ can be obtained from (2.15) by setting formally $n_B\equiv n_*+1=0$, which gives

$$-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\theta}\log\left[\frac{P^+(\theta)}{R_0(\theta)}\right] = \frac{2\lambda}{\pi}\int_{-\infty}^{+\infty}\mathrm{d}x\,\cos\left(\frac{2\lambda\theta x}{\pi}\right)\frac{\sinh(\lambda+2\xi/\pi)x}{2\sinh x\cosh\lambda x}. \tag{2.19}$$

Note that if $2\xi/\pi>1$, the integral in (2.19) diverges. Finally, we complete this list by the following two expressions:

$$\begin{aligned}-\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\theta}\log\left[\frac{P^\pm_{\delta_{N,n}}(\theta)}{R_0(\theta)}\right] &= -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\theta}\log\left[\frac{P^\pm_{\beta_n}(\theta)}{R_0(\theta)}\right]+\frac{2\lambda}{\pi}\int_{-\infty}^{+\infty}\mathrm{d}x\,\cos\left(\frac{2\lambda\theta x}{\pi}\right)\\ &\quad\times\left[\frac{\sinh((2\xi/\pi)-2n_*-2)x}{\sinh x}-\frac{2\cosh x\sinh((2\xi/\pi)+2N-\lambda-1)x}{2\sinh x\cosh\lambda x}\right].\end{aligned} \tag{2.20}$$

For the integral representation of R_0 see [6].

3. Exact solution of the regularized boundary sine–Gordon model

3.1. The XXZ chain with boundary magnetic field

The XXZ model in a boundary magnetic field

$$\mathcal{H}=\frac{\pi-\gamma}{2\pi\sin\gamma}\left[\sum_{i=1}^{L-1}(\sigma_i^x\sigma_{i+1}^x+\sigma_i^y\sigma_{i+1}^y+\Delta(\sigma_i^z\sigma_{i+1}^z-1))+h(\sigma_1^z-1)+h'(\sigma_L^z-1)\right] \quad (3.1)$$

was discussed in [11], where its eigenstates were constructed using the Bethe ansatz. As usual, these eigenstates $\mathcal{H}|n\rangle=E|n\rangle$ are linear combinations of the states with n down spins, located at $x_1,\dots,x_n$ on the chain

$$|n\rangle=\sum f^{(n)}(x_1,\dots,x_n)|x_1,\dots,x_n\rangle.$$

Consider for simplicity the case $n=1$. The wavefunction $f^{(1)}(x)$ reads [11]:

$$\begin{aligned}f^{(1)}(x)&=[\mathrm{e}^{-\mathrm{i}k}+(h'-\Delta)]\mathrm{e}^{-\mathrm{i}(L-x)k}-(k\to-k)\\&=\left[\frac{\sinh\frac{1}{2}(\mathrm{i}\gamma+\alpha)}{\sinh\frac{1}{2}(\mathrm{i}\gamma-\alpha)}\right]^{L-x}\frac{\sin\gamma\sinh\frac{1}{2}(\alpha+\mathrm{i}\gamma H')}{\sinh\frac{1}{2}(\mathrm{i}\gamma-\alpha)\sin\frac{1}{2}(\gamma+\gamma H')}-(\alpha\to-\alpha)\end{aligned} \quad (3.2)$$

where we defined the new variables as in [11]: $\Delta=-\cos\gamma$, $k=f(\alpha,\gamma)$,

$$\gamma H=f(\mathrm{i}\gamma,-\mathrm{i}\ln(h-\Delta))=-\gamma-\mathrm{i}\ln\frac{h-\mathrm{i}\sin\gamma}{h+\mathrm{i}\sin\gamma} \quad (3.3)$$

(and similarly for H'), and

$$f(a,b)=-\mathrm{i}\ln\left[\frac{\sinh\frac{1}{2}(\mathrm{i}b-a)}{\sinh\frac{1}{2}(\mathrm{i}b+a)}\right]. \quad (3.4)$$

When h varies from 0 to $+\infty$, γH increases monotonically from $-\pi-\gamma$ to $-\gamma$ according to (3.3) if we take the main branch of the logarithm.

Denote $h_{\mathrm{th}}=1-\cos\gamma$. This 'threshold' value of h corresponds to $\gamma H=-\pi$; its meaning will become clear below. When h varies from $-\infty$ to 0, γH increases monotonically from $-\gamma$ to $\pi-\gamma$. For the purposes of the present work we confine our attention to the region h, $h'>0$ and choose $\gamma\in(0,\frac{\pi}{2})$. Other regions in the parameter space can be obtained using the discrete symmetries of the Hamiltonian (3.1): $\sigma^z\to-\sigma^z$ on each site or on the odd sites only. The parameter k in (3.2) is not arbitrary, but satisfies the Bethe equation [11]:

$$\mathrm{e}^{\mathrm{i}(2L-2)k}\frac{(\mathrm{e}^{\mathrm{i}k}+h-\Delta)(\mathrm{e}^{\mathrm{i}k}+h'-\Delta)}{(\mathrm{e}^{-\mathrm{i}k}+h-\Delta)(\mathrm{e}^{-\mathrm{i}k}+h'-\Delta)}=1 \quad (3.5)$$

or

$$\left[\frac{\sinh\frac{1}{2}(\alpha-\mathrm{i}\gamma)}{\sinh\frac{1}{2}(\alpha+\mathrm{i}\gamma)}\right]^{2L}\frac{\sinh\frac{1}{2}(\alpha-\mathrm{i}\gamma H)\sinh\frac{1}{2}(\alpha-\mathrm{i}\gamma H')}{\sinh\frac{1}{2}(\alpha+\mathrm{i}\gamma H)\sinh\frac{1}{2}(\alpha+\mathrm{i}\gamma H')}=1. \quad (3.6)$$

Note that the wavefunction (3.2) depends on H implicitly through the solution of the Bethe equation (3.6) $\alpha(H,H')$. Besides, one can multiply the amplitude (3.2) by any overall scalar factor depending on α, L, H, and H'. The Bethe equations in the sector of arbitrary $n>1$ can be found in [11].

3.2. The Bethe equations for the inhomogeneous XXZ chain

The real object of interest for us is actually the inhomogeneous six-vertex model with boundary magnetic field on an open strip. The inhomogeneous six-vertex model is obtained by giving an alternating imaginary part $\pm\mathrm{i}\Lambda$ to the spectral parameter on alternating vertices of the six-vertex model [12, 13]. It was argued in [6], generalizing known results for the periodic case [13] that this model on an open strip provides, in the scaling limit Λ, $L \to \infty$, lattice spacing$\to 0$, a lattice regularization of (1.1) with $\beta^2 = 8\gamma$ and a value of φ_0 at the boundary related to the magnetic field. The reader can find more details on the model in the references; it is actually closely related to the XXZ chain we discussed above. In particular, the wavefunction can be expressed in terms of the roots α_j of the Bethe equations [11, 12]

$$\left[\frac{\sinh\frac{1}{2}(\alpha_j+\Lambda-\mathrm{i}\gamma)}{\sinh\frac{1}{2}(\alpha_j+\Lambda+\mathrm{i}\gamma)}\frac{\sinh\frac{1}{2}(\alpha_j-\Lambda-\mathrm{i}\gamma)}{\sinh\frac{1}{2}(\alpha_j-\Lambda+\mathrm{i}\gamma)}\right]^L\frac{\sinh\frac{1}{2}(\alpha_j-\mathrm{i}\gamma H)}{\sinh\frac{1}{2}(\alpha_j+\mathrm{i}\gamma H)}\frac{\sinh\frac{1}{2}(\alpha_j-\mathrm{i}\gamma H')}{\sinh\frac{1}{2}(\alpha_j+\mathrm{i}\gamma H')}$$
$$=\prod_{m\neq j}\frac{\sinh\frac{1}{2}(\alpha_j-\alpha_m-2\mathrm{i}\gamma)}{\sinh\frac{1}{2}(\alpha_j-\alpha_m+2\mathrm{i}\gamma)}\frac{\sinh\frac{1}{2}(\alpha_j+\alpha_m-2\mathrm{i}\gamma)}{\sinh\frac{1}{2}(\alpha_j+\alpha_m+2\mathrm{i}\gamma)}. \tag{3.7}$$

By construction of the Bethe-ansatz wavefunction, $\mathrm{Re}\,\alpha_j > 0$. Note that the solutions of (3.7) $\alpha_j = 0$, $\mathrm{i}\pi$ should be excluded because the wavefunction vanishes identically in this case. The analysis of solutions of (3.7) is very similar to the case of the XXZ chain in a boundary magnetic field. We consider the regime $0 < \gamma < \pi/2$, which falls into the attractive regime $0 < \beta^2 < 4\pi$ in the sine–Gordon model (1.1). We set hereafter $\gamma = \pi/t$ and for technical simplicity restrict t to be a positive integer. In the limit $L \to \infty$ this constraint implies that in the bulk only the strings of length from 1 to $t-1$ are allowed, along with the anti-strings.

Taking the logarithm of equation (3.7), one obtains

$$L[f(\alpha_j+\Lambda,\gamma)+f(\alpha_j-\Lambda,\gamma)]+f(\alpha_j,\gamma H)+f(\alpha_j,\gamma H')$$
$$=2\pi l_j+\sum_{m\neq j}[f(\alpha_j-\alpha_m,2\gamma)+f(\alpha_j+\alpha_m,2\gamma)] \tag{3.8}$$

where l_j is an integer. We also recall the formula for the eigenenergy associated with the roots α_j [11, 12],

$$E=\frac{\pi-\gamma}{\pi}\sum_{\alpha_j}[f'(\alpha_j+\Lambda,\gamma)+f'(\alpha_j-\Lambda,\gamma)]. \tag{3.9}$$

3.3. Thirring model with boundary

Since the bulk sine–Gordon model is a bosonized version of the bulk massive Thirring model, one can expect that the boundary sine–Gordon model is a bosonized version of the Thirring model with certain boundary conditions. The quickest way to identify this boundary Thirring model is to use the Bethe ansatz equations (3.7). Writing the most general $U(1)$-invariant boundary interaction,

$$H_{\mathrm{T}}=\int_0^L\mathrm{d}x\,[-\mathrm{i}\psi_1^+\psi_{1x}+\mathrm{i}\psi_2^+\psi_{2x}+m_0\psi_1^+\psi_2+m_0\psi_2^+\psi_1+2g_0\psi_1^+\psi_2^+\psi_2\psi_1]$$
$$+\sum_{ij}a_{ij}\psi_i^+\psi_j(0)+\sum_{ij}a'_{ij}\psi_i^+\psi_j(L). \tag{3.10}$$

The entries of the 2×2 matrices $\mathbf{A} = \{a_{ij}\}$, $\mathbf{A}' = \{a'_{ij}\}$ can be determined up to one arbitrary parameter ϕ by the hermicity of H_T and the consistency of the boundary conditions ($\det \mathbf{A} = 0$). For the left boundary, the matrix $\mathbf{A}$ looks like

$$\mathbf{A} = \frac{1}{2\sin\phi}\begin{pmatrix} e^{-i\phi} & 1 \\ 1 & e^{i\phi} \end{pmatrix} \tag{3.11}$$

and the boundary condition reads $\psi_1(0) = -e^{i\phi}\psi_2(0)$ (and similarly for the right boundary).

To find the eigenvectors of the Hamiltonian (3.10), $H_T\Psi = E\Psi$, one can use the same wavefunctions as for the bulk Thirring model [14], and modify them by analogy with the example of an XXZ chain in a boundary magnetic field [11]. This way one gets the equations for the set of rapidities α_j:

$$\begin{aligned} e^{2im_0L\sinh\alpha_j} &= \frac{\cosh\frac{1}{2}(\alpha_j + i\phi)\cosh\frac{1}{2}(\alpha_j + i\phi')}{\cosh\frac{1}{2}(\alpha_j - i\phi)\cosh\frac{1}{2}(\alpha_j - i\phi')} \\ &\quad \times \prod_{m\neq j} \frac{\sinh\frac{1}{2}(\alpha_j - \alpha_m - 2i\gamma)}{\sinh\frac{1}{2}(\alpha_j - \alpha_m + 2i\gamma)} \frac{\sinh\frac{1}{2}(\alpha_j + \alpha_m - 2i\gamma)}{\sinh\frac{1}{2}(\alpha_j + \alpha_m + 2i\gamma)} \end{aligned} \tag{3.12}$$

where γ is related to g_0 in the usual way [14]. These equations look quite similar to (3.7). The mapping can be made complete by taking the limit $\Lambda \to \infty$ in (3.7) with the identification $m_0 = 4e^{-\Lambda}\sin\gamma$.

The derivation of these equations is rather cumbersome, therefore to illustrate the procedure we comment on the simplest case of a one-particle sector, which is nevertheless sufficient to obtain the form of the boundary terms in (3.12). We make an ansatz $\Psi = \int_0^L dy\, \chi^\lambda(y)\psi_\lambda^+(y)|0\rangle$, where λ is the spinor index, $\chi(y)$ is the wavefunction, and $|0\rangle$ is the unphysical vacuum annihilated by ψ_λ.

The equation $H_T\Psi = E\Psi$ reduces to

$$-i\sigma_3\frac{\partial}{\partial x}\chi + m_0\sigma_1\chi + \mathbf{A}\chi\delta(x) + \mathbf{A}'\chi\delta(x-L) = E\chi. \tag{3.13}$$

where $\boldsymbol{\sigma}_i$ are the Pauli matrices. We look for the solution of (3.13) in the form

$$\begin{pmatrix} \chi_1 \\ \chi_2 \end{pmatrix} = a(\alpha)\begin{pmatrix} e^{-\alpha/2} \\ e^{\alpha/2} \end{pmatrix} e^{im_0x\sinh\alpha} - a(-\alpha)\begin{pmatrix} e^{\alpha/2} \\ e^{-\alpha/2} \end{pmatrix} e^{-im_0x\sinh\alpha}. \tag{3.14}$$

Substituting it into (3.13) we get $E = m_0\cosh\alpha$ and, besides, two boundary conditions to be solved. The first one, at $x = 0$, determines the form of the factor $a(\alpha) = \cosh\frac{1}{2}(\alpha - i\phi)$, while the second one, at $x = L$, gives rise to the Bethe equation

$$e^{2im_0L\sinh\alpha} = \frac{\cosh\frac{1}{2}(\alpha + i\phi)\cosh\frac{1}{2}(\alpha + i\phi')}{\cosh\frac{1}{2}(\alpha - i\phi)\cosh\frac{1}{2}(\alpha - i\phi')}$$

which determines α. Comparing the Bethe equation (3.7) with (3.12), and using the relation between ξ and H obtained below in section 5, we find the relation between the boundary parameters ϕ and φ_0 in the Hamiltonians (3.10) and (1.1) respectively:

$$\phi = \beta\varphi_0 - \beta^2/8.$$

Thus, the integrable boundary condition for the $U(1)$-invariant boundary Thirring model equivalent to (1.1) reads:

$$\psi_2(0) = -e^{i\beta^2/8 - i\beta\varphi_0}\psi_1(0). \tag{3.15}$$

It would be interesting to obtain the result (3.15) directly from the Hamiltonian (1.1) using an extension of the Coleman–Mandelstam bosonization technique to the case with boundary. However, to our knowledge such an extension has not been developed yet. The naive application of the known Coleman–Mandelstam 'bulk' formulae does not give the factor $e^{i\beta^2/8}$ in (3.15), which seems to be some kind of 'boundary anomaly' (see [23] for the related discussion).

4. Solutions of the Bethe ansatz equations with boundary terms

As is well known in the case of the bulk Thirring model or equivalently the periodic XXZ chain, the bound states are associated with various types of solutions of the Bethe ansatz equations involving, in general, complex roots [14]. By analogy, we expect the boundary bound states to correspond to new complex solutions made possible by the boundary terms: technically, these terms change the asymptotic behaviour of the left-hand side of the Bethe ansatz equations (3.7), allowing solutions different from the usual strings, and 'rooted' on a H dependent basic root.

Consider first the simplest example of the XXZ chain with one spin down as given in section 3.1 and (3.6). Since our goal is to study purely boundary effects, we will look for the solutions of the Bethe equations that give rise to a wavefunction localized at $x = 0$ or $x = L$ and exponentially decreasing away from the boundary. The states described by such wavefunctions will be referred to as the 'boundary bound states' below. For this, one should have α purely imaginary in (3.2). We consider here the limit of L large, when the left and the right boundaries can be treated independently and the overlap of the corresponding wavefunctions is negligibly small (for the physical applications it is necessary to take the scaling limit anyway). In the limit $L \to \infty$, it is easy to check that there are two such solutions to (3.6): $\alpha = i\alpha_0 = -i\gamma H + i\varepsilon(L, H, H')$ and $\alpha = i\alpha_0' = -i\gamma H' + i\varepsilon'(L, H, H')$, where $\varepsilon \sim \exp(-2\kappa L)$ and we defined $\kappa > 0$ as

$$e^{-\kappa} = \left|\frac{\sin\frac{1}{2}(-\gamma H - \gamma)}{\sin\frac{1}{2}(-\gamma H + \gamma)}\right|$$

(similar relations are assumed for ε', κ'). Solution α_0' gives a wavefunction (3.2) localized at $x = L$: $f^{(1)}(x) \sim e^{-\kappa'(L-x)}$. Solution α_0 gives a wavefunction localized at $x = 0$, $f^{(1)}(x) \sim e^{-\kappa x}$, provided we renormalize the wavefunction (3.2):

$$f^{(1)} \to f^{(1)}[\sinh\tfrac{1}{2}(\alpha - i\gamma H)\sinh\tfrac{1}{2}(\alpha + i\gamma H)]^{1/2}. \tag{4.1}$$

In the special case $H = H'$, there is only one proper solution $\alpha = i\alpha_0 = -i\gamma H + i\varepsilon(L, H)$ with $\varepsilon \sim \exp(-\kappa L)$. The wavefunction (3.2) behaves as the superposition of the 'left' and the 'right' boundary bound states, $f^{(1)} \sim (e^{-\kappa x} + e^{-\kappa(L-x)})$. Note that the boundary bound state appears in the above example only when the boundary magnetic field is large enough: namely, $h > h_{th}$†. This follows from the fact that α should be such that $0 < \alpha_0 < \pi$.

Now consider the general case of the inhomogeneous model (3.7). The basic boundary one-string solution to (3.7) is still $\alpha = i\alpha_0 = -i\gamma H + i\epsilon$, provided that $0 < \alpha_0 < \pi$.

† More generally, the criterion of existence of boundary bound state solutions allows us to determine threshold fields for any Δ. For this, let us examine (3.5). The parameter k is defined modulo 2π, therefore we restrict k to lie within $k \in (0, 2\pi)$. Two possibilities, $k = ia$ and $k = \pi + ia$ where $a > 0$, lead to two different threshold fields, determined by the fact that the denominator in (3.5) should vanish:

$$h_{th}^{(1)} = \Delta + 1 \qquad h_{th}^{(2)} = \Delta - 1$$

and the regions where boundary bound states could in principle exist are $h < \Delta - 1$ and $h > \Delta + 1$ (one has to be careful here and check that these solutions of BE indeed correspond to the *stable* states of the model). When $\Delta > 1$, there are two different threshold fields, in agreement with the results of Jimbo *et al* [15]. In the region of interest, $|\Delta| < 1$, there is only one threshold field $h_{th}^{(1)}$.

This solution is possible due to an argument very similar to the one used in the bulk: as $L \to \infty$, the two first terms of (3.7) decrease exponentially fast, while the third increases exponentially fast, and $\epsilon \sim \exp(-2\kappa L)$ with

$$e^{-\kappa} = \frac{\sinh^2(\Lambda/2) + \sin^2((\alpha_0 - \gamma)/2)}{\sinh^2(\Lambda/2) + \sin^2((\alpha_0 + \gamma)/2)}. \tag{4.2}$$

Recall that for the bulk problem when there is no boundary term, the right-hand side of (3.7) would have to decrease exponentially, forcing the existence of a 'partner' root at $\alpha - 2i\gamma$.

One can construct similarly boundary n-strings which consist of the points $i\alpha_0, i\alpha_0 + 2i\gamma, \ldots, i\alpha_0 + 2i(n-1)\gamma$ (see figure 1). By convention, $n = 0$ means there is no boundary string, i.e. all complex solutions are in the usual bulk strings. The possible values of n are restricted by the fact that the upper point of the complex should be below $i\pi$: $\max(n) = [(\pi - \alpha_0)/2\gamma] + 1$, where the square bracket denotes the integer part. To show that the boundary n-string is indeed a solution to (3.7), we introduce infinitesimal corrections ε_i to the positions of the points of the complex [16]. Taking the modulus of both sides of (3.7) with $\alpha_j = i\alpha_0 + 2ik\gamma$ and multiplying equations for $k = 0, \ldots, n-1$ we obtain $\exp\{-2L(\kappa_1 + \kappa_2 + \cdots + \kappa_n)\} \sim \varepsilon_1$, where ε_1 denotes the correction to the point $i\alpha_0$. The behaviour of the remaining ε_k follows from ε_1 by recursion. For example, for the two-string ε_2 is given by $|\varepsilon_1 - \varepsilon_2| \sim \exp(-2L\kappa_2)$†.

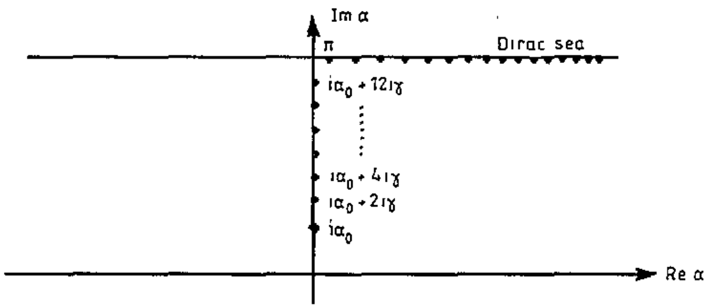

Figure 1. The first type of boundary string. In the ground state the boundary string of maximum allowed length is occupied.

Additional boundary strings can be obtained by adding the roots $i\alpha_s$ below $i\alpha_0$ so that $i\alpha_s = i\alpha_0 - 2is\gamma$, with $s = 1, 2, \ldots, N$ (see figure 2). Together with the boundary n-string above α_0, they form the complex which we call the boundary (n, N)-string. To analyse the existence of such complexes as the solutions of (3.7) we introduce as before the infinitesimal corrections ε_s to the roots α_s, where now $s = n, n-1, \ldots, 1, -1, -2, \ldots, -N$. Then, equations (3.7) with $\alpha_j = i\alpha_s$ tell us that the range of N should be

$$\frac{\alpha_0}{2\gamma} < N < \frac{\pi + \alpha_0}{2\gamma}. \tag{4.3}$$

In other words, the inequality (4.3) states that the lowest root of the boundary string should be below the axis $\mathrm{Im}\,\alpha = 0$ and above the axis $\mathrm{Im}\,\alpha = -\pi$. Another constraint follows

† Note that, associated with each boundary n-string, there is also the solution to (3.7) obtained by complex conjugation of all α's. The existence of such a 'mirror image' is the consequence of the symmetry of equations (3.7) and it is of no importance to physics. In the bulk case, it is easy to show [17] that all solutions are invariant under complex conjugation, but this result does not hold here. In fact, a solution which has both the boundary n-string and its mirror image would lead to a vanishing wavefunction.

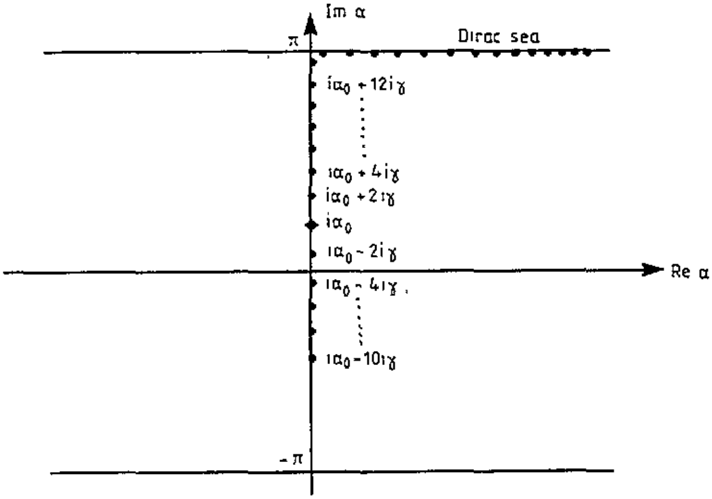

Figure 2. The second type of boundary string.

if we multiply equations (3.7) for all the roots of the boundary (n, N)-string. This gives $\exp(-2L\sum\kappa_s) = \varepsilon_1$. So, one should have $\sum\kappa_s > 0$. The latter sum can be easily evaluated if one uses expression (4.2) simplified in the limit $\Lambda \to \infty$: $\kappa = 4\mathrm{e}^{-\Lambda}\sin\gamma\sin\alpha_0$. The constraint obtained in such a way forces the number of roots above the $\mathrm{Im}\,\alpha = 0$ axis in the boundary string to be greater than the number of roots below $\mathrm{Im}\,\alpha = 0$.

We have not been able to find any reasonable additional solution to the Bethe ansatz equations. The two sets of boundary strings we have encountered appear to be in one-to-one correspondence with the boundary bound states identified in section 2 using the bootstrap approach. To clarify this identification we now compute related masses and S-matrices.

5. S-matrices and bound state properties from the exact solution

5.1. *Bare and physical Bethe ansatz equations*

The 'bare' Bethe equations follow from taking the derivative of (3.8). Defining $2L(\rho_k + \rho_k^h)\mathrm{d}\alpha$ to be the number of roots in the interval $\mathrm{d}\alpha$, one obtains coupled integral equations for the densities of strings $\rho_1, \ldots, \rho_{t-1}$ and anti-strings ρ_a:

$$
\begin{aligned}
2\pi(\rho_k + \rho_k^h) &= \tfrac{1}{2}p_k' - f_{ka}' * \rho_a - \sum_{l=1}^{t-1} f_{kl}' * \rho + \frac{1}{2L}\left(u_k - \omega f_{n;k}'^{(L)} - \omega' f_{n';k}'^{(R)}\right) \\
2\pi(\rho_a + \rho_a^h) &= -\tfrac{1}{2}p_a' + f' * \rho_a + \sum_{l=1}^{t-1} f_{al}' * \rho_l + \frac{1}{2L}\left(u_a + \omega f_n'^{(L)} + \omega' f_{n'}'^{(R)}\right)
\end{aligned}
\qquad (5.1)
$$

where $*$ denotes convolution:

$$
f * g(\alpha) = \int_{-\infty}^{\infty} \mathrm{d}\beta\, f(\alpha - \beta) g(\beta).
$$

These densities are originally defined for $\alpha > 0$. But the equations allow us to define $\rho_k(-\alpha) \equiv \rho_k(\alpha)$ in order to rewrite the integrals to go from $-\infty$ to ∞. If we totally neglect the boundary terms (terms$\sim L^{-1}$) in (5.1), we will end up with the same equations as for the periodic inhomogeneous six-vertex model [12]. The various kernels and sources in (5.1) are defined as follows:

$$p_a(\alpha) = f(\mathrm{i}\pi + \alpha + \Lambda, \gamma) + f(\mathrm{i}\pi + \alpha - \Lambda, \gamma)$$
$$p_k(\alpha) = \sum_{\alpha_i} f(\alpha_i + \Lambda, \gamma) + f(\alpha_i - \Lambda, \gamma)$$

where the sum in the last expression is taken over the rapidities of the bulk k-string roots centred on α.

The kernels f_{kl} are the phase shifts of the bulk k-string on the bulk l-string obtained by summing (3.4) over the rapidities of string roots. The boundary terms are:

$$u_a(\alpha) = -2f'(2\alpha, 2\gamma) - f'(\alpha + \mathrm{i}\pi, \gamma H) - f'(\alpha + \mathrm{i}\pi, \gamma H') - 2\pi\delta(\alpha)$$
$$u_k(\alpha) = \sum_{\alpha_i} [2f'(2\alpha_i, 2\gamma) + f'(\alpha_i, \gamma H) + f'(\alpha_i, \gamma H')] - 2\pi\delta(\alpha)$$

(the sum above is over the roots of the bulk k-string centred on α),

$$f_n^{(L,R)}(\alpha) = \sum_{\alpha_i} f(\mathrm{i}\pi + \alpha - \alpha_i, 2\gamma) + f(\mathrm{i}\pi + \alpha + \alpha_i, 2\gamma)$$

and α_i denotes the rapidities of the roots in the boundary n-string.

$$f_{n;k}^{(L,R)}(\alpha) = \sum_{\alpha_i} \sum_{\alpha_j} f(\alpha_j - \alpha_i, 2\gamma) + f(\alpha_j + \alpha_i, 2\gamma)$$

where α_i denotes the roots in the boundary n-string, while α_j denotes the roots in the bulk k-string centred on α. The parameters ω, ω' are equal to 1 or 0, depending on whether the boundary string is present or not. In our ferromagnetic case the ground state of the periodic inhomogeneous XXZ chain is filled with anti-strings. The physical Bethe equations are obtained [18, 19] by eliminating the 'non-physical' density ρ_a from the right-hand side of (5.1). This is done simply by solving for ρ_a in the last equation in (5.1) and substituting it into the others. The result is

$$\begin{aligned} 2\pi(\rho_k + \rho_k^h) &= \tfrac{1}{2}p_k' + \tfrac{1}{2}\frac{f_{ak}'}{2\pi - f'} * p_a' + \frac{f_{ak}'}{2\pi - f'} * 2\pi\rho_a^h \\ &\quad - \sum_{l=1}^{t-1} \left(f_{kl}' + \frac{f_{ak}' f_{al}'}{2\pi - f'} \right) * \rho_l + \frac{1}{2L} U_{n,n';k} \\ 2\pi(\rho_a + \rho_a^h) &= -\tfrac{1}{2}\frac{2\pi p_a'}{2\pi - f'} - \frac{f'}{2\pi - f'} * 2\pi\rho_a^h + \sum_{l=1}^{t-1} \frac{f_{al}'}{2\pi - f'} * 2\pi\rho_l + \frac{1}{2L} U_{n,n';a} \end{aligned} \tag{5.2}$$

where

$$U_{n,n';a} = 2\pi \frac{u_a + \omega f_n'^{(L)} + \omega' f_{n'}'^{(R)}}{2\pi - f'} \tag{5.3}$$

$$U_{n,n';k} = u_k - \omega f_{n;k}'^{(L)} - \omega' f_{n';k}'^{(R)} - f_{ak}' * U_{n,n';a}/2\pi \tag{5.4}$$

and different products (ratios) of kernels are defined through their Fourier transforms.

5.2. *The mass spectrum of boundary bound states*

We assume first that the ground state is built by filling up the Dirac sea with anti-strings, as in the case of the periodic XXZ chain. We will see below that this is not always true. The presence of the boundary strings in the Bethe equations deforms the distribution of roots and modifies the density of the Dirac sea ρ_a by a term $\delta\rho_a/2L$ of order L^{-1}. With the boundary n-string, the Bethe equation for the density of the Dirac sea particles $\tilde{\rho}_a$ is

$$\frac{1}{2}p_a'(\alpha) - \frac{1}{2L}u_a(\alpha) = \int_{-\infty}^{+\infty} f'(\alpha-\beta)\tilde{\rho}_a(\beta)\,\mathrm{d}\beta - 2\pi\tilde{\rho}_a(\alpha) + \frac{1}{2L}f_n'(\alpha) \tag{5.5}$$

where f_n was defined above. Subtracting from (5.5) the equation for the density of the Dirac sea alone,

$$\frac{1}{2}p_a'(\alpha) - \frac{1}{2L}u_a(\alpha) = \int_{-\infty}^{+\infty} f'(\alpha-\beta)\rho_a(\beta)\,\mathrm{d}\beta - 2\pi\rho_a(\alpha) \tag{5.6}$$

one obtains the equation for $\delta\rho_a$:

$$\begin{aligned} 0 &= -\int_{-\infty}^{+\infty} f'(\alpha-\beta)\delta\rho_a(\beta)\,\mathrm{d}\beta + 2\pi\delta\rho_a(\alpha) - f_n'(\alpha) \\ \delta\rho_a &\equiv 2L(\tilde{\rho}_a - \rho_a). \end{aligned} \tag{5.7}$$

The solution to (5.7) can be written in terms of the Fourier transform $\delta\hat{\rho}_a(k) = \int \mathrm{d}\alpha\,\mathrm{e}^{ik\alpha}\delta\rho_a(\alpha)$ as follows:

$$\delta\hat{\rho}_a(k) = \frac{\hat{f}_n'(k)}{2\pi - \hat{f}'(k)}. \tag{5.8}$$

From the boundary n-string $\mathrm{i}\alpha_0 + 2\mathrm{i}\gamma s$, $s = 0, 1, \ldots, n-1$, we obtain

$$\hat{f}_n'(k) = -2\pi\frac{4\cosh\gamma k\sinh n\gamma k\cosh(\alpha_0+\gamma n-\gamma)k}{\sinh\pi k} \tag{5.9}$$

$$\delta\hat{\rho}_a(k) = -\frac{2\cosh\gamma k\sinh n\gamma k\cosh(\alpha_0+\gamma n-\gamma)k}{\sinh\gamma k\cosh(\pi-\gamma)k} \tag{5.10}$$

where we use

$$\hat{f}'(k) = 2\pi\frac{\sinh(\pi-2\gamma)k}{\sinh\pi k}.$$

Expressions (5.9) and (5.10) are valid for the n-strings with $n = 1, 2, \ldots, [(t+H)/2]$. For the longest n-string with $n = [(t+H)/2]+1 \equiv n_* + 1$ the Fourier transforms $\hat{f'}_n, \delta\hat{\rho}_a$ differ from (5.9) and (5.10):

$$\begin{aligned} \hat{f'}_*(k) = 2\pi&\frac{2\sinh(\pi-2\gamma)k\cosh(\alpha_0+2\gamma n_*-\pi)k}{\sinh\pi k} \\ &-2\pi\frac{4\cosh\gamma k\sinh n_*\gamma k\cosh(\alpha_0+\gamma n_*-\gamma)k}{\sinh\pi k} \end{aligned} \tag{5.11}$$

$$\begin{aligned} \delta\hat{\rho}_a(k) = &\frac{\sinh(\pi-2\gamma)k\cosh(\alpha_0+2\gamma n_*-\pi)k}{\sinh\gamma k\cosh(\pi-\gamma)k} \\ &-\frac{2\cosh\gamma k\sinh n_*\gamma k\cosh(\alpha_0+\gamma n_*-\gamma)k}{\sinh\gamma k\cosh(\pi-\gamma)k}. \end{aligned} \tag{5.12}$$

The conserved $U(1)$ charge in the boundary XXZ chain is the total projection of the spin on the z axis. In the thermodynamic limit the charge of the boundary n-string with respect to the vacuum is determined by [11]:

$$Q_n = n + \int_0^{+\infty} 2L\tilde{\rho}_a\,\mathrm{d}\alpha - \int_0^{+\infty} 2L\rho_a\,\mathrm{d}\alpha = n + \tfrac{1}{2}\int_{-\infty}^{+\infty}\delta\rho_a\,\mathrm{d}\alpha = n + \tfrac{1}{2}\delta\hat{\rho}_a(0). \tag{5.13}$$

Using (5.10) we obtain for the n-string $Q_n = 0$, and for the longest boundary string equation (5.12) yields $Q_* = \pi/2\gamma$. Similarly, the mass of the boundary strings in the thermodynamic limit according to (3.9) is given by

$$m_n = h_n + \int_0^{+\infty} 2L\tilde{\rho}_a h_a \,d\alpha - \int_0^{+\infty} 2L\rho_a h_a \,d\alpha = h_n + \tfrac{1}{2}\int_{-\infty}^{+\infty} h_a \delta\rho_a \,d\alpha \tag{5.14}$$

where the expression for h_a is

$$\hat{h}_a(k) = \frac{\pi - \gamma}{\pi}\hat{p}'_a = -2(\pi - \gamma)\frac{2\sinh\gamma k \cos\Lambda k}{\sinh\pi k}$$

and the soliton mass [12]

$$m = 2\exp\left[\frac{-\Lambda\pi}{2(\pi - \gamma)}\right].$$

We obtain in the limit $\Lambda \to \infty$

$$m_n = m\left[\sin\frac{\pi}{2\lambda}(2n - 1 - H) + \sin\frac{\pi}{2\lambda}(H + 1)\right] \tag{5.15}$$

$$m_* = m\sin\frac{\pi}{2\lambda}(H + 1). \tag{5.16}$$

Since the parameter H varies in the interval $-\lambda - 1 < H < -1$, the mass of the longest string m_* (5.16) is always negative, while the other boundary strings have positive masses (5.15). This means that the vacuum we have been working with is an unstable one in the region $-t < H < -1$ ($h > h_{\mathrm{th}}$). To cure the situation we define a new correct ground state by attributing the longest boundary string to the Dirac sea. The boundary excitations are obtained by successive removing of particles from the top of the longest boundary string. The charge and mass of such excitations with respect to the correct ground state are given by

$$Q_n = -\frac{\pi}{2\gamma} \qquad m_n = m\cos\frac{\pi}{2\lambda}(\lambda + 1 + H - 2n) \qquad n = 0, 1, \ldots, n_*. \tag{5.17}$$

Note that the number of excitations (5.17) is equal to the number of roots in the longest boundary string, $n_* + 1$. The charge of such boundary excitations is equal to the charge of the hole in the Dirac sea. We identify a hole with a sine–Gordon soliton, and the boundary excitations described above, with the boundary bound states $|\beta_n\rangle$ (2.3). Their masses and charge (5.17) and the counting coincide provided that

$$t + H + 1 = \frac{2\xi}{\pi} \tag{5.18}$$

and the lattice charge Q is properly normalized. This expression is in fact valid for all values of $h > 0$. The authors of [6], deriving this relation in the region $h < h_{\mathrm{th}}$, obtained a different expression because they used a different branch of logarithm in (3.3).

In the above discussion we considered the boundary bound states related to one of the boundaries (say, the left one). In principle, one should include in the ground state the longest boundary string $\mathrm{i}\alpha'_0 + 2\mathrm{i}\gamma l$, $l = 0, 1, \ldots, [(t + H')/2]$, corresponding to the right boundary as well. The energy of the excitations due to both boundary strings is a superposition of energies of the form (5.17). When $H = H'$, these two boundary strings overlap and the usual Bethe wavefunction vanishes. However, on physical grounds we do not expect anything special to happen when the boundaries are identical. So, in such a case, one should use as a wavefunction a properly renormalized version of the limit $H \to H'$ of the usual Bethe wavefunction.

When the magnetic field varies, the above picture indicates a qualitative change in the structure of the ground state at values $H = -t, -t+2, -t+4, \ldots$. At these values, the mass of the bound state with the highest mass approaches the soliton mass and it becomes unstable. As discussed in [20] and [4] for the Ising case, this decay corresponds to large boundary fluctuations that propagate deeply into the bulk.

The mass of the boundary (n, N)-string with respect to the correct vacuum can be calculated analogously. The result is

$$m_{n,N} = m\cos\left(\frac{\xi}{\lambda} - \frac{\pi}{2\lambda}\right) + m\cos\left(\frac{\xi}{\lambda} - \frac{2n+1}{2\lambda}\pi\right) - m\cos\left(\frac{\xi}{\lambda} + \frac{2N-1}{2\lambda}\pi\right) \tag{5.19}$$

where we used (5.18) to express H in terms of ξ. This result is rather confusing to us, because the above mass does not correspond in general to one of the bound state masses found in the bootstrap approach. It can be considered as a sum of such masses, hinting that the (n, N)-string describes actually coexisting bound states, but the calculation of the corresponding boundary S-matrix does not allow such an interpretation. We are forced (but see the conclusion) to consider that only the (n, N)-strings with $n = n_* + 1$ occur, that is the physical excitations are built by adding roots to the ground-state configuration below $\mathrm{i}\alpha_0$. The charge and energy of such excitations with respect to the correct vacuum is given by

$$Q_N = \frac{\pi}{2\gamma} - \frac{\pi}{2\gamma} = 0 \qquad m_N = m\cos\left(\frac{\xi}{\lambda} - \frac{\pi}{2\lambda}\right) - m\cos\left(\frac{\xi}{\lambda} + \frac{2N-1}{2\lambda}\pi\right). \tag{5.20}$$

These coincide with the charge and mass of the boundary bound states $|\delta_{n=0,N}\rangle$ (2.10). The range of N (4.3) agrees with the range of the corresponding parameter in (2.9).

5.3. Boundary S-matrices

It remains to check that the boundary S-matrices obtained by the bootstrap approach coincide with those of the lattice model. To extract the boundary S-matrices from the Bethe equations we will follow the discussion of [6]. Briefly, the idea of the method is as follows. The physical excitations of the lattice model in the limit $\Lambda \to \infty$ can be thought of as relativistic quasi-particles with rapidities θ_i. The integrability implies that the set $\{\theta_i\}$ is conserved. Moreover, if the scattering matrices are diagonal, each particle preserves its rapidity. Assuming that this is the case, the quantization of a gas of $\mathcal{N}$ quasi-particles on an interval of length L results in the integral equations for the set of allowed rapidities [6]:

$$2\pi(\rho_b + \rho_b^h) = m_b\cosh\theta + \sum_{c=1}^{p}\varphi_{bc} * \rho_c + \frac{1}{2L}\Theta_b \tag{5.21}$$

where the subscript stands for the type of particle, and

$$\begin{aligned} \varphi_{bc}(\theta) &= -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\theta}\ln S_{bc}(\theta) \\ \Theta_b(\theta) &= -\mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\theta}\ln R_{\beta}^{b(L)}(\theta) - \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\theta}\ln R_{\beta'}^{b(R)}(\theta) + \mathrm{i}\frac{\mathrm{d}}{\mathrm{d}\theta}\ln S_{bb}(2\theta) - 2\pi\delta(\theta). \end{aligned} \tag{5.22}$$

Equations (5.21) should be compared with the physical BE (5.2), which gives bulk and boundary S-matrices. We will confine our attention to the boundary S-matrices only, keeping track of those terms in (5.3), (5.4), and (5.22), which depend on the boundary magnetic field (the field-independent terms contribute to R_0 and their agreement has been shown in [6]). The discussion for the left boundary is completely parallel to that of the right one. Also, it is sufficient to consider only b = soliton and b = anti-soliton in (5.21). We identify a hole in the anti-string distribution in (5.2) with a soliton in (5.21), and $(t-1)$-string with an anti-soliton. Below we give explicit expressions only for the kernels in (5.2) which we need for our analysis. The other expressions are listed in [6].

Suppose first that $h < h_{\text{th}}$ $(-t-1 < H < -t)$. This corresponds to the case without boundary excitations in the spectrum, $\xi < \pi/2$. Choose $\omega = \omega' = 0$ in (5.2). Then

$$\hat{u}_a^{(L)}(k) = 2\pi \frac{\sinh(2\pi + \gamma H)k}{\sinh \pi k} + \cdots$$

(we omit the H-independent and H'-dependent terms),

$$\hat{U}_a^{(L)} = 2\pi \frac{\sinh(2\pi + \gamma H)k}{2\sinh \gamma k \cosh(\pi - \gamma)k} + \cdots.$$

Using (5.22) we compare this expression with (2.19) (recall that the rapidity α should be renormalized $\alpha \to \theta = t\alpha/2\lambda$) and find complete agreement under the identification (5.18). Similarly, one can use

$$\hat{u}_{t-1}^{(L)} = -2\pi \frac{\sinh(\pi + \gamma H)k \sinh(\pi - \gamma)k}{\sinh \pi k \sinh \gamma k} + \cdots$$
$$\hat{U}_{t-1}^{(L)} = -2\pi \frac{\sinh(2 + H)\gamma k}{2\sinh \gamma k \cosh(\pi - \gamma)k} + \cdots$$

to compare U_{t-1} with (2.18) and obtain agreement as well.

Next, suppose that $h > h_{\text{th}}$ $(-t < H < -1)$. To obtain the boundary S-matrices for scattering on the ground state $|0\rangle_B$ set $\omega = \omega' = 1$ and choose the boundary string to be the longest string, $n = n_* + 1$ in (5.2). Then, using (5.11), $\hat{f'}_{n_*+1;t-1} = -\hat{f}'_*$ and

$$\hat{u}_a^{(L)} = 2\pi \frac{\sinh \gamma H k}{\sinh \pi k} + \cdots$$
$$\hat{u}_{t-1}^{(L)} = \hat{u}_a^{(L)} - 2\pi \frac{\sinh(H + 2[(1-H)/2])\gamma k}{\sinh \gamma k} + \cdots \tag{5.23}$$

we obtain

$$\frac{\hat{U}_{n_*+1;a}^{(L)}}{2\pi} = \frac{\sinh \gamma H k}{2\sinh \gamma k \cosh(\pi - \gamma)k} + \frac{\sinh(\pi - 2\gamma)k \cosh(H + t - 2n_*)\gamma k}{\sinh \gamma k \cosh(\pi - \gamma)k}$$
$$- \frac{2\cosh \gamma k \sinh n_* \gamma k \cosh(H - n_* + 1)\gamma k}{\sinh \gamma k \cosh(\pi - \gamma)k} + \cdots$$
$$\hat{U}_{n_*+1;t-1}^{(L)} = \hat{U}_{n_*+1;a}^{(L)} - 2\pi \frac{\sinh(H + 2[(1-H)/2])\gamma k}{\sinh \gamma k} + \cdots$$

which agrees with (2.15) and (2.18) under the identification (5.18). Note that the last relation, which follows directly from (5.3), (5.4), and (5.23), is valid also for $\hat{U}_{n;a}$ and $\hat{U}_{n;t-1}$ with any n. In the same manner one can calculate the boundary S-matrices for scattering on the boundary n-strings and check that they indeed coincide with (2.16) and (2.17) under the condition (5.18). For this, one needs to take $\omega = \omega' = 1$ in (5.2) and use (5.9), $\hat{f}'_{n;t-1} = -\hat{f}'_n$. Finally, one can compute also the boundary S-matrix for the scattering on the $(n_* + 1, N)$-strings, again in agreement with the bootstrap results.

6. Conclusion

The question of boundary bound states even in the simple Dirichlet case appears rather frustrating: using the XXZ lattice regularization or equivalently the Thirring model, we have only been able to recover the β_n and $\delta_{n=0,N}$ boundary bound states. A way out is to consider solutions of the Bethe ansatz equations made of an (n, N)-string superposed with the (n_*+1)-string that describes the ground state. This is not allowed in principle, in the model we consider, because the Bethe wavefunction vanishes when two roots are equal. However, putting formally such a solution in the equations gives the masses of the $\delta_{n,N}$ states and the S-matrix also matching the bootstrap results! But the meaning of this is not clear to us.

Finally, let us mention that one can calculate the ground-state energy in the thermodynamic limit by solving equation (5.5) for the ground-state density and using (3.9):

$$E_{\mathrm{gr}} = \int_0^{+\infty} 2L\tilde{\rho}_a(\alpha)h(\alpha)\,\mathrm{d}\alpha.$$

As a result, we get the combination $E_{\mathrm{gr}} = E_{\mathrm{bulk}} + E_{\mathrm{boundary}}$, where E_{bulk} is the well known sine–Gordon ground-state energy [21]

$$E_{\mathrm{bulk}} = -\frac{Lm^2}{4}\tan\frac{\pi\gamma}{2(\pi-\gamma)}$$

and E_{boundary} is the contribution of the boundary terms ($\Lambda \to \infty$):

$$E_{\mathrm{boundary}} = \frac{m}{2}\left[\frac{\sin[(H+2)\gamma\pi/2(\pi-\gamma)]}{\sin[\pi^2/2(\pi-\gamma)]} + 1 + \cot\frac{\pi^2}{4(\pi-\gamma)}\right].$$

We see that the ground-state energy of the boundary sine–Gordon model is a smooth function of the boundary magnetic field for the whole range of h in the XXZ regularization, hence of φ_0. The changes in ground-state structure do not affect E, as is expected since in such a unitary model there is no (one-dimensional) boundary transition.

The finite size corrections to the ground-state energy themselves (the genuine Casimir effect) can be computed using the technique developed in [21]. It is also interesting to consider the inhomogeneous six-vertex model with an imaginary boundary magnetic field ensuring commutation with $U_q sl(2)$ [22]. This should presumably lead to a solution of minimal models with integrable boundary conditions. We will report on these questions separately [24].

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References

[1] Fendley P, Lesage F and Saleur H 1995 *J. Stat. Phys.* **79** 799
[2] Guinea F, Hakim V and Muramatsu A 1985 *Phys. Rev. Lett.* **54** 263
[3] Kane C and Fisher M 1992 *Phys. Rev.* B **46** 15 233
[4] Ghoshal S and Zamolodchikov A 1994 *Int. J. Mod. Phys.* A **9** 3841
[5] Ghoshal S 1994 *Int. J. Mod. Phys.* A **9** 4801
[6] Fendley P and Saleur H 1994 *Nucl. Phys.* B **428** 681
[7] Grisaru M T, Mezincescu L and Nepomechie R Direct calculation of the boundary S matrix for the open Heisenberg chain, hep-th/9407089
[8] Saleur H, Skorik S and Warner N P 1995 *Nucl. Phys.* B **441** 421
[9] Zamolodchikov A B and Zamolodchikov Al B 1980 *Ann. Phys.* **120** 253
[10] Dashen R, Hasslacher B and Neveu A 1975 *Phys. Rev.* D **11** 3424

[11] Alcaraz F, Barber M, Batchelor M, Baxter R and Quispel G 1987 *J. Phys. A: Math. Gen.* **20** 6397
[12] Reshetikhin N and Sale H (B) colur 1994 *Nucl. Phys.* B **419** 507
[13] Destri C and de Vega H 1989 *J. Phys. A: Math. Gen.* **22** 1329
[14] Korepin V E 1979 *Theor. Math. Phys.* **41** 953
Bergknoff H and Thacker H 1979 *Phys. Rev.* D **19** 3666
[15] Jimbo M, Kedem R, Tojima T, Konno H and Miwa T 1995 *Nucl. Phys.* B **441** 437
[16] Gaudin M *La Fonction d'Onde de Bethe* (Paris: Masson)
[17] Babelon O, de Vega H J and Viallet C M 1983 *Nucl. Phys.* B **220** 13
[18] Destri C and Lowenstein J H 1982 *Nucl. Phys.* B **205** 369
[19] Kirillov A N and Reshetikhin N 1987 *J. Phys. A: Math. Gen.* **20** 1565, 1587
[20] McCoy B and Wu T 1973 *The Two-Dimensional Ising Model* (Cambridge, MA: Harvard University Press)
[21] Destri C and de Vega H J 1995 *Nucl. Phys.* B **438** 413
[22] Pasquier V and Saleur H 1990 *Nucl. Phys.* B **330** 523
[23] Ameduri M, Konik R and LeClair A Boundary sine–Gordon interactions at the free fermion point *Preprint* CLNS 95/1321, hep-th/9503227
[24] LeClair A, Mussardo G, Saleur H and Skorik S Boundary energy and boundary states in integrable quantum field theories *Preprint* hep-th/9503227